Key

Math 4 Name_______________

5-6 **Using Derivatives to Analyze Graphs** Date_______

In this Activity, you will be working towards the following learning goals:
I can use derivatives to analyze functions.

The following passages come from a calculus textbook and address the characteristics of derivatives we have discussed previously as well as some new ones. ON YOUR OWN, read the following paragraphs and try to answer the questions immediately following. We will stop to discuss them as a class. *Remember, next year you may not always understand everything that your professor explains – this is what you will be left with to help you figure it out. Start practicing now!*

*Note: The ◀ in the text means the author is suggesting that you "check this fact yourself" – meaning it is a good place to stop and assess if you understand what you are reading!!

The Geometry of Derivatives

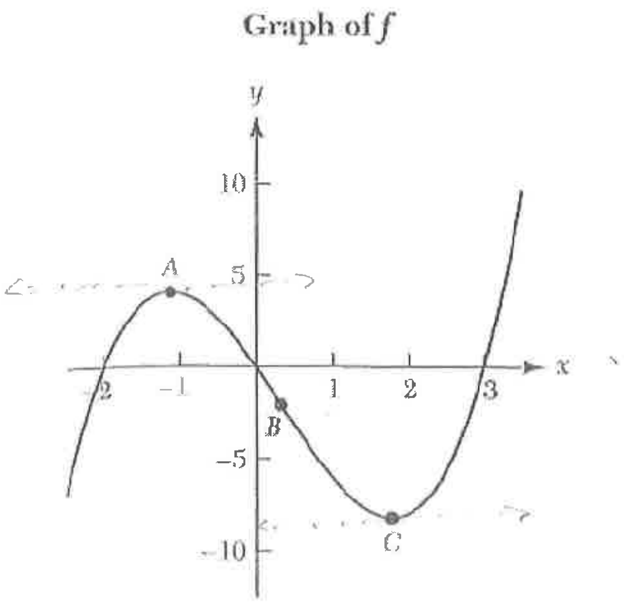
The geometric relationship between a function f and its derivative f' is easy to state:

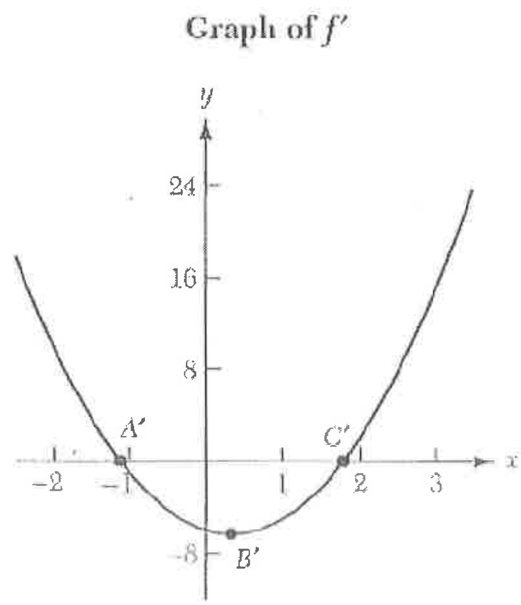
> *For any input a, $f'(a)$ is the slope of the line tangent to the f-graph at $x = a$.*

This statement—innocuous as it appears—is one of the most important in this book. The next two sections (and much of this book) explore its meaning and implications.

Graphs of f and f': An Extended Example

Graphs of a function f and its derivative f' follow. (For the moment, no formulas are given—or needed. We'll return to these functions symbolically at the end of the next section.) ◀ Three interesting points are labeled on each. ◀





Based on our prior learning, why are the points labeled on each graph "interesting"? How are the points A and C related to A' and C'? We will learn about the relationship between B and B' in just a wee bit . . .

A+C = max/min of f + where f' is 0

B is where f is decreasing the most which is why it is also the min. of f.

The f'-graph tells how slopes of tangent lines to the f-graph behave. At B, for instance, the f-graph seems ► to have a slope around -6; for this reason, the f'-graph has height -6 at B'.

First let's observe several straightforward geometric relationships between the two graphs, introducing some useful new terminology in the process.

Why does the graph of f' have a height of -6 when the graph of f has a tangent line with a slope of -6?

f' gives us the slopes of the tangent lines to f.

The Sign of f' The graph of f rises to the left of A, falls between A and C, and rises again to the right of C. The sign of $f'(x)$ tells whether the line tangent to the f-graph at x points up or down. At A' and C', ► f' changes sign. Thus, at A and C, f itself changes direction.

Why does f change directions when f' changes sign? Explain in terms of the meaning of a derivative.

These points are where the derivative (the slope of f) goes from positive to negative or vice versa. Slope only changes sign when your graph goes from incr. to decr. or vice versa.

Stationary, Maximum, and Minimum Points The points A and C, with approximate coordinates $(-1.1, 4)$ and $(1.8, -8)$, ► where the f-graph is horizontal, are obviously of interest. They mark, respectively, high and low points of the graph. The situation looks clear, but to avoid later confusion, it will pay to be extremely picky with language here—especially about the distinction between inputs to f (called points) and outputs from f (called values).

Here, in full detail, is the situation at $A \approx (-1.1, 4)$. The domain point $x = -1.1$ is called a local maximum point of f; the corresponding output—$f(-1.1) \approx 4$—is called a local maximum value of f. At C ► the situation is similar: $x = 1.8$ is a **local minimum point** of f, and $f(1.8) \approx -8$ is the corresponding **local minimum value** of f. The x-coordinates of both A and C are called **stationary points** of f. ◄ (We say local rather than global because elsewhere in its domain f may assume larger or smaller values.) The corresponding points A' and C' occur where the f'-graph crosses the x-axis (i.e., at roots of f').

"zeroes"

What is the difference between a **local minimum point** *and a* **local minimum value**? *Why do you think the author makes a point of being "picky" with the vocabulary?*

Point: the x-value at a certain location
Value: the y-value at that location.

It's important to know if we are talking about x or y-values.

Concavity and Inflection The point B, near $x = 0.3$, is an **inflection point** of f: At B, the f-graph's **direction of concavity** changes; from concave down to concave up. ◄ The point B has another special geometric property: At B, the graph of f points most steeply downward.

The corresponding point on the f'-graph, B', is easier to see; it's a local minimum point. Later we'll use this property and some algebra to find the *exact* location of B. ◄

One informal way to explain concavity is to think of concave up as where the graph "holds water" and concave down as where the graph "spills water". Give a definition of concave up and concave down in your own words.

Concave Down

Concave Up (cup)

What f' Says about f

Interpreting the derivative function f' in terms of the slopes of tangent lines on the f-graph has many important geometric implications. We summarize several below.

Increasing or Decreasing?

A function f **increases** where its graph rises ◄ and **decreases** where its graph falls. The following definition captures these natural ideas in analytic language.

Definition: Let I denote the interval (a,b).

A function is **increasing** on I if $f(x_1) < f(x_2)$ whenever $a < x_1 < x_2 < b$

A function is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $a < x_1 < x_2 < b$

Explain what $a < x_1 < x_2 < b$ *means.*

The x-values from $a \to b$ are increasing

From smallest to largest: a, x_1, x_2, b.

Explain what $f(x_1) < f(x_2)$ and $f(x_1) > f(x_2)$ means.

$f(x_1) \rightarrow$ y-value at x_1

$f(x_2) \rightarrow$ y-value at x_2

$f(x_1)$ smaller than $f(x_2)$ Opposite!

In your own words, what does it mean for a graph to be increasing? For a graph to be decreasing?

Increasing: As x-values increase, so do y-values.

Decreasing: As x-values increase, y-values decrease.

> **Fact** If $f'(x) > 0$ for all x in I, then f increases on I. If $f'(x) < 0$ for all x in I, then f decreases on I.

★Positive derivative means increasing original graph★

This fact certainly *sounds* reasonable. To say that $f'(x) > 0$ means that the tangent line to the f-graph at x points upward. With any luck, so should the f-graph itself. Racetrack ▶ language makes the fact seem even simpler: If dog f has positive velocity, then dog f always runs *forward*.

Behavior at a Point. Functions are often said to increase or decrease *at a point*. To say, for example, that f increases *at* $x = 3$ means that f increases on some interval—perhaps a small one—containing $x = 3$. Now we can restate the preceding fact ▶ as follows:

> **Fact** If $f'(a) > 0$, then f is increasing at $x = a$. If $f'(a) < 0$, then f is decreasing at $x = a$.

At $x = 1$, for instance, $f'(1) < 0$. The Fact says—and the picture agrees—that at $x = 1$ the f-graph is decreasing.

The converse of the above fact is as follows

> **Fact** If f increases at $x = a$, then $f'(a) \geq 0$; if f decreases at $x = a$, then $f'(a) \leq 0$.

Explain in your own words what the previous two "Facts" tell us about the relationship between the graph of a function and the graph of the functions derivative.

When the derivative is positive, f increases.

When f' is negative, f decreases.

Converse is also true: when f increases, f' is positive.
when f decreases, f' is negative.

Finding Maximum and Minimum Points

Geometric intuition says that at a local maximum point or local minimum point, a smooth graph must be "flat." ▶ More succinctly:

> **Fact** On a smooth graph every local maximum or local minimum point x_0 is a stationary point—i.e., a root of f'. Root = Zero = x-intercept

This fact has immense practical value. To find maximum and minimum values of f, the fact says, we can limit our search to roots of f'. Each such root is a stationary point and therefore, possibly, a maximum or minimum point (but not a sure thing—a stationary point may be only a "flat spot" in the graph). ▶

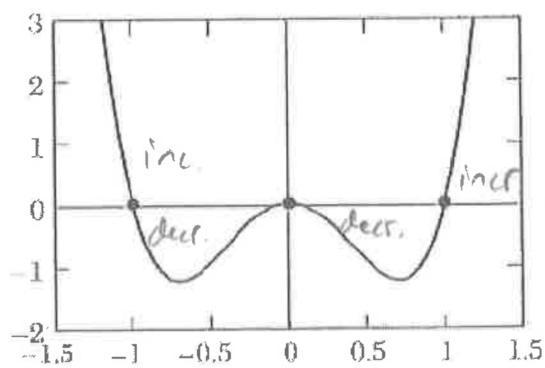
The next example—an important one—shows how to sort out all the possibilities.

Explain what the above "Fact" means in terms of finding local maximum/minimum points of graphs using the calculus we've learned.

f has a local min or max when f' has an x-interzept. These occur at the same x-value within their respective graph.

EXAMPLE 2 The graph of a function f' appears as follows; the f-graph is not shown (for now). Three points of interest are bulleted. Where, if anywhere, does f have local maximum or local minimum points? Why?

Graph of f'



What are the local minimum or maximum points? This is the same type of question that was on our last quiz.

X = −1, 0, 1

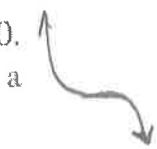
Specifically, Max at x = −1, min at x = 1, "flat spot" at x = 0.

Solution The three bullets on the graph—at $x = -1$, $x = 0$, and $x = 1$—represent roots of f' and therefore correspond to stationary points of f. What type of stationary point is each one: a local maximum, a local minimum, or just a flat spot? The key to deciding is to check the sign of f' just before and just after each stationary point. We take each root of f' in turn.

At $x = -1$ Just before (i.e., to the left of) $x = -1$, $f'(x) > 0$. Therefore (by an earlier Fact), f increases until $x = -1$. Just after $x = -1$, $f'(x) < 0$, so f decreases immediately after $x = -1$. This means that f has a local maximum at $x = -1$.

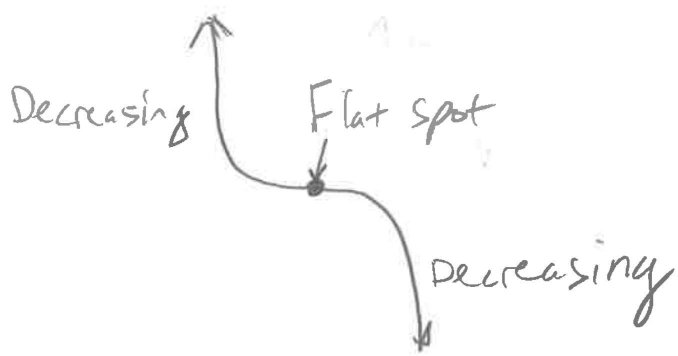
At $x = 1$ Consider values of x near $x = 1$. The graph shows that if $x < 1$, $f'(x) < 0$; if $x > 1$, $f'(x) > 0$. Thus, reasoning as above, f decreases before $x = 1$ and increases after $x = 1$. This means that f has a local minimum at $x = 1$.

At $x = 0$ This time the graph shows that $f'(x) < 0$ on both sides of $x = 0$. Thus, f must decrease before *and* after $x = 0$, so $x = 0$ is neither a maximum nor a minimum point, but just a flat spot in the f-graph.



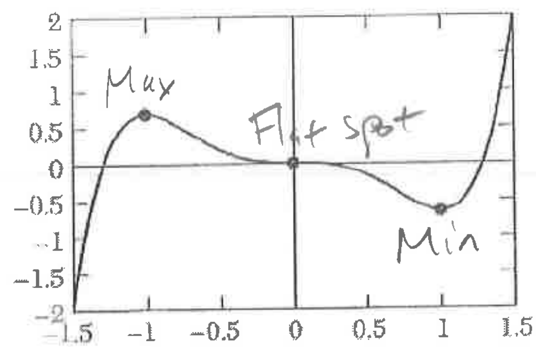
Note above the explanation for why f does NOT have a local max/min at $x = 0$!! This is more in depth than we have looked in the past. Then explain, in your own words, what is happening to the graph of f at $x = 0$.

There is a "kink" in the graph at $x=0$.



Here, at last, is a possible f-graph. It agrees with everything we said.

Graph of f



Let's summarize what we know about stationary points.

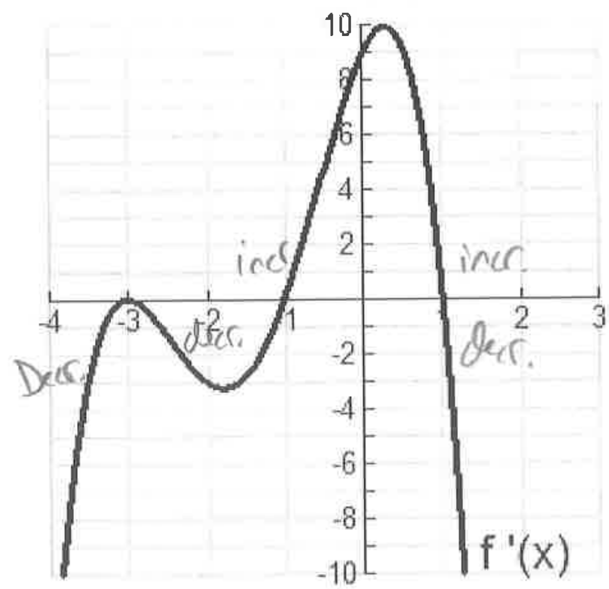
> **Fact** (First Derivative Test) Suppose that $f'(x_0) = 0$.
>
> - If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, then x_0 is a local minimum point.
> - If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then x_0 is a local maximum point.

Practice: Given the below graph of $f'(x)$, find all the local maximum and minimum points of $f(x)$ x-values

Max: X=-1 incr. decr.

Min: X=-1 decr. incr.

Flat spot: X=-3



~~Check your answer with Mr. Sheppard before you move on to the next page!!~~

Concave Up or Concave Down?

So far we've described concavity and inflection points informally, in graphical language. Here's a more formal, analytic definition:

Definition: The graph of f is **concave up** at $x = a$ if the derivative function f' is increasing at $x = a$.
The graph of f is **concave down** at $x = a$ if f' is decreasing at $x = a$.
Any point at which a graph's direction of concavity changes is called an **inflection point**.

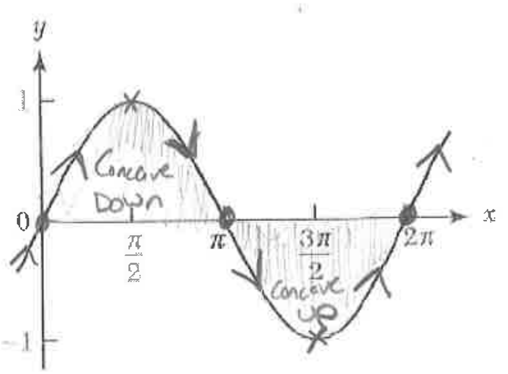
Finding Inflection Points from the Graph of f′

The direction of concavity of the graph of f depends, as the definitions show, on whether f' increases or decreases. An inflection point occurs wherever f' *changes direction*—i.e., wherever f' has a local minimum or a local maximum.

Example: When you take a calculus class, you will learn that the derivative of the sine function is the cosine function. That is, if $f(x) = \sin x$, then $f'(x) = \cos x$. Based on this knowledge, discuss the concavity of the sine function. Find all inflection points and describe them in derivative language.

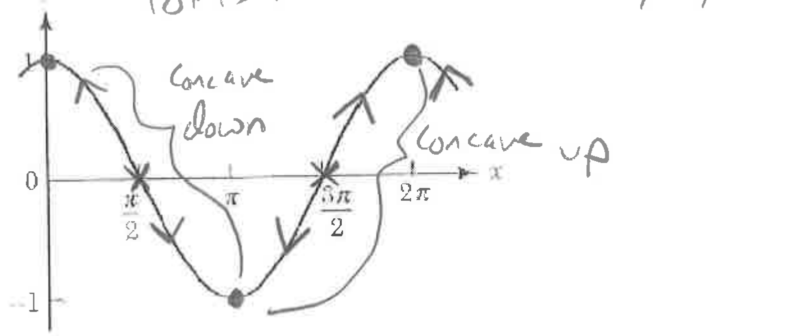
Solution: Note the graphs of $f(x) = \sin x$ and $f'(x) = \cos x$ below.

Graph of $f(x) = \sin x$



Graph of $f'(x) = \cos x$

Points of inflection: $x = 0, \pi, 2\pi$



Notice:

Stationary Points f has stationary points (a local maximum point and a local minimum point) $x = \pi/2$ and at $x = 3\pi/2$—exactly the roots of f'.

Increasing or Decreasing? f increases on the intervals $(0, \pi/2)$ and $(3\pi/2, 2\pi)$; on the same intervals, f' is positive.

Concavity f is concave down on $(0, \pi)$—where f' decreases—and concave *up* on $(\pi, 2\pi)$—where f' increases. In fact, f illustrates every possible combination of increasing/decreasing and concavity behavior.

Inflection Points f has an inflection point at each multiple of π—precisely where f' assumes a local maximum or local minimum. ■

Annotate (make notes on) the graphs above that illustrate the stationary points, increasing and decreasing intervals, concavity, and inflection points.

If f' has a local maximum or minimum, what is the value of f''?

$f''=0$

Based on your answer to the above question, how can you find inflection points using the second derivative?

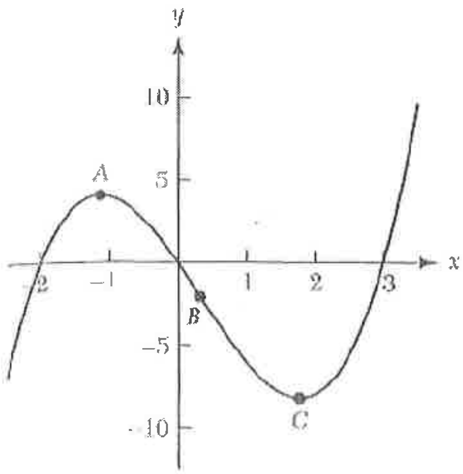
Find the x-values that make the second derivative equal zero.

What do your inflection points tell you about the graph of f?

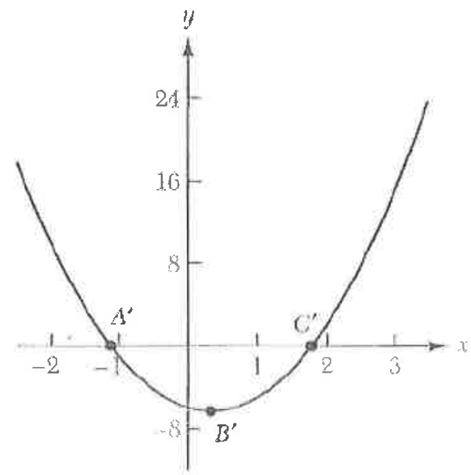
They tell you where the graph goes from concave up to concave down + vice versa.

Practice: The graphs of f and f' are reprinted below. They again illustrate the definition of concavity given above.

Graph of f



Graph of f'



Explain, referencing the labeled points on the above graphs, how the graph of f' illustrate the stationary points, increasing and decreasing intervals, concavity, and inflection points of the graph of f.

Stationary: A' + C'

Increasing: Before A' + after C'
$X < A'$ + $X > C'$

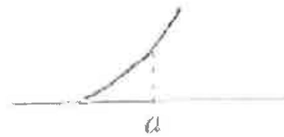
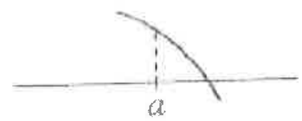
Decreasing: Between A' + C'
$A' < X < C'$

Inflection Points: B'

Concave Down: $X < B'$
(Derivative is decreasing)

Concave up: $X > B'$
(Derivative is increasing)

Fill in the below table based on what you have learned about the first and second derivative:

Conditions on the Derivatives	**Description of** $f(x)$ at $x=a$	**Graph of** $y=f(x)$ near $x=a$
1. $f'(a)$ is positive $f''(a)$ is positive	$f(x)$ is increasing $f(x)$ is concave up	
2. $f'(a)$ is positive $f''(a)$ is negative	$f(x)$ is increasing $f(x)$ is concave down	
3. $f'(a)$ is negative $f''(a)$ is positive	$f(x)$ is decreasing $f(x)$ is concave up	
4. $f'(a)$ is negative $f''(a)$ is negative	$f(x)$ is decreasing $f(x)$ is concave down	

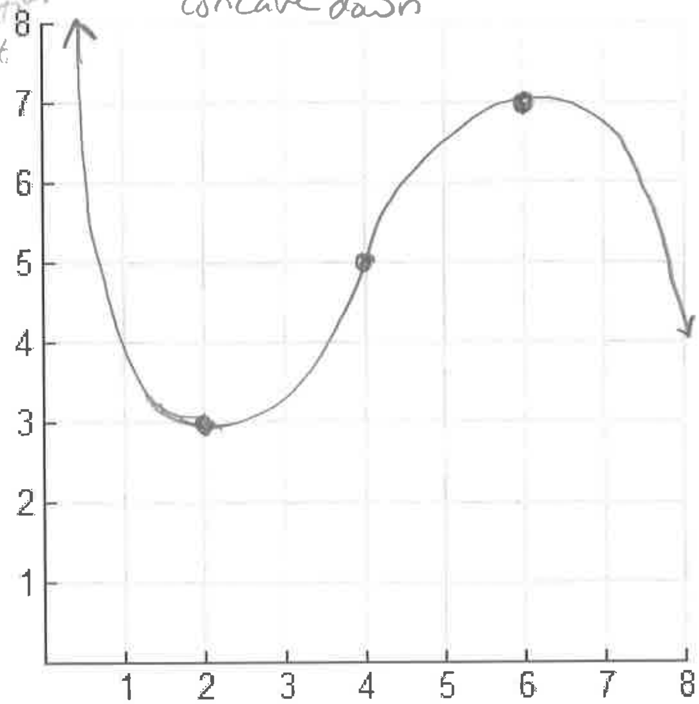
~~Check your answer with Mr. Sheppard before you move on!!~~

Practice: Sketch a graph of a function $f(x)$ with all the following properties:

a. $(2,3)$, $(4,5)$, and $(6,7)$ are on the graph.

b. $f'(6)=0$ and $f'(2)=0$ → max/min

c. $\underbrace{f''(x)>0 \text{ for } x<4}_{\text{concave up}}$, $\underbrace{f''(4)=0}_{\text{inflection point}}$, and $\underbrace{f''(x)<0 \text{ for } x>4}_{\text{concave down}}$.



Write a "rule" for how the first derivative is related to the graph of $f(x)$.

If $f' > 0$, then f is increasing (+ the other way around)

If $f' < 0$, then f is decreasing (+ the other way around)

If $f' = 0$, then f has a min/max or flat spot @ that x-value.
(+ other way around)

Write a "rule" for how the second derivative is related to the graph of $f(x)$.

$f'' > 0$ if and only if f is concave up.

$f'' < 0$ if and only if is concave down.

$f'' = 0$ if and only if f has an inflection point at that x-value.

Practice: Sketch a graph of the below function. Label all relative/absolute maximums and minimums, points of inflection, and the y-intercept. Use your calculator ONLY FOR COMPUTATIONS No graphing! Use derivatives and show all work below.

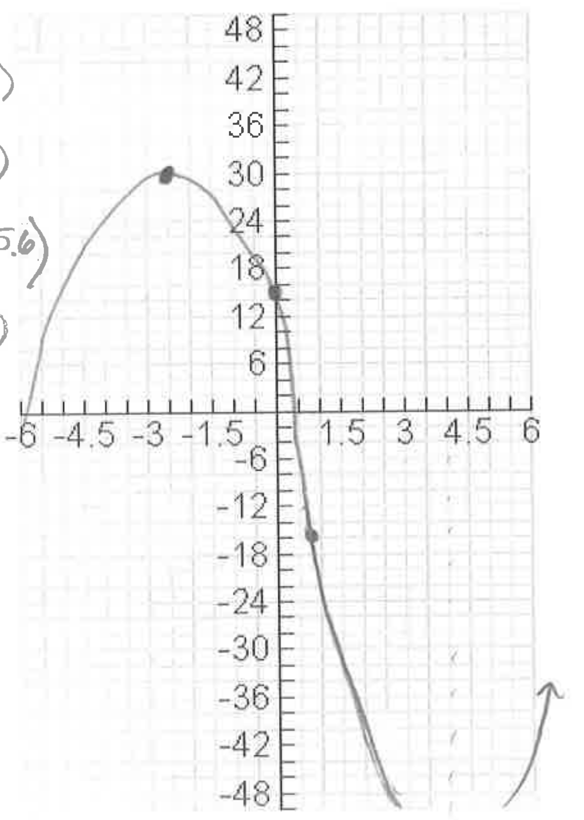
$$f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 20x + 15$$

Max: $(-5/2, 30.2)$
Min: $(4, -61.\bar{3})$
Inflection: $(\frac{3}{4}, -15.6)$
y-int: $(0, 15)$

$f'(x) = 2x^2 - 3x - 20$
$= (2x+5)(x-4)$
$x = -\frac{5}{2}$ $\quad x = 4$

$2x+5$	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
	$+$	$-$	$+$
	Incr.	Decr.	Incr.

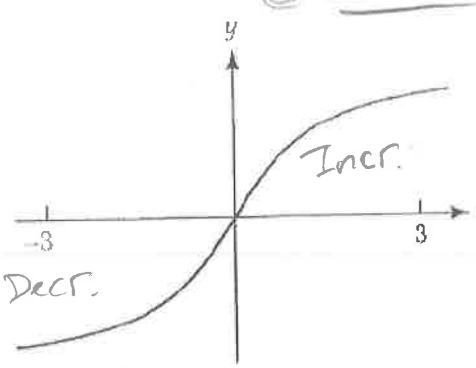
$-\frac{5}{2}$ Max $\quad$ 4 Min



$f''(x) = 4x - 3$ $\quad 0 = 4x - 3$ $\quad 3 = 4x$ $\quad x = \frac{3}{4} \rightarrow$ inflection point

$4x-3$: Concave down $-$ | $\frac{3}{4}$ | Concave up $+$

Practice (more difficult): The graph of a derivative of the function f is shown



→ x-intercepts

a. The equation $f(x) = 0$ can have no more than two solutions on the interval $[-3,3]$. Explain why.

f will only change direction once
Goes from decreasing to increasing
Example:

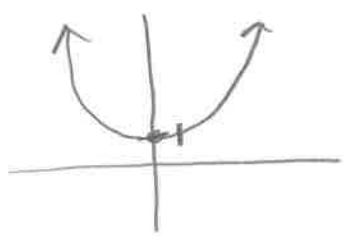
b. Explain why f cannot have two zeros in the interval $[0,3]$

The graph of f is increasing over that entire interval, except at x=0 which is a minimum point.

X Y

c. Suppose that $f(0) = 1$. How many solutions does the equation $f(x) = 0$ have on the interval $[-3,3]$. Explain.

None. The minimum value of f occurs at x=0 since this is where the derivative is 0 ($f'(0) = 0$). So the graph of f would look like the figure below:



Solution Because f' is negative on the interval $(-3, 0)$ and positive on the interval $(0, 3)$, f itself must decrease on $(-3, 0)$ and increase on $(0, 3)$. Thus f has a local minimum at $x = 0$ and no other stationary points. Therefore:

a. The graph of f is U-shaped, so it can intersect the interval $[-3, 3]$ in at most two places.

b. The graph of f increases on the interval $(0, 3)$, so it can cross the x-axis no more than once on that interval.

c. If $f(0) = 1$, then f has no roots in the interval $[-3, 3]$, because f has its minimum value at $x = 0$. ■